

SEMISTABLE PRINCIPAL G -BUNDLES IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let X be a normal projective variety defined over an algebraically closed field k of positive characteristic. Let G be a connected reductive group defined over k . We prove that some Frobenius pull back of a principal G -bundle admits the canonical reduction E_P such that its extension by $P \rightarrow P/R_u(P)$ is strongly semistable (see Theorem 5.1).

Then we show that there is only a small difference between semistability of a principal G -bundle and semistability of its Frobenius pull back (see Theorem 6.3). This and the boundedness of the family of semistable torsion free sheaves imply the boundedness of semistable (rational) principal G -bundles.

0. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic. Let X be a normal projective variety over k with a very ample divisor H . One can then define the degree $\deg E$ of a torsion free sheaf E on X with respect to H and its slope $\mu(E) = \deg E / \operatorname{rk} E$. We say that a torsion free sheaf E on X is *slope semistable* with respect to H if for every subsheaf F of E we have $\mu(F) \leq \mu(E)$. Every torsion free sheaf has a canonical filtration with semistable quotients, the so called *Harder–Narasimhan filtration*. Let $\mu_{\max}(E)$ denote the slope of the first factor of this filtration, i.e., the slope of the maximal destabilizing subsheaf of E .

Assume that $\operatorname{char} k = p$ and let $F: X \rightarrow X$ be the Frobenius morphism. It is well known that if E is semistable then its Frobenius pull back F^*E need not longer be semistable. If all the Frobenius pull backs $(F^k)^*E$ are semistable then E is called *strongly semistable*. By the Ramanan–Ramanathan theorem (see Theorem 2.7) such sheaves are well behaved under tensor operations. In particular, a torsion free part of the tensor product of two strongly semistable torsion free sheaves is strongly semistable.

In [La1] the author proved that for every torsion free sheaf E there exists some non-negative integer l such that the factors of the Harder–Narasimhan filtration of $(F^l)^*E$ are strongly semistable. Therefore if E_1 and E_2 are torsion free sheaves there exists some l such that

$$\mu_{\max}((F^l)^*(E_1 \otimes E_2)) = \mu_{\max}((F^l)^*E_1) + \mu_{\max}((F^l)^*E_2).$$

On the other hand one can easily see that

$$\mu_{\max}(E_1 \otimes E_2) \leq \frac{\mu_{\max}((F^l)^*(E_1 \otimes E_2))}{p^l}.$$

One can also show that the differences

$$\frac{\mu_{\max}((F^l)^*E_1)}{p^l} - \mu_{\max}(E_1) \quad \text{and} \quad \frac{\mu_{\max}((F^l)^*E_2)}{p^l} - \mu_{\max}(E_2)$$

are non-negative and bounded from the above by some explicit numbers depending only on (X, H) , p and the ranks of E_1 and E_2 (see Theorem 1.3). Therefore we get a precise bound on the degree of instability of the tensor product with respect to the degree of instability of E_1 and E_2 . This implies, e.g., that for large primes p the tensor product of semistable sheaves is semistable.

The main aim of this paper is to develop an analogue of the above approach in the case of principal G -bundles, or more generally, rational G -bundles.

The first step in the above approach is to prove that some Frobenius pull back of a principal G -bundle E admits the strongly semistable canonical reduction (see Theorem 5.1). This is proved using Behrend's combinatorial method allowing to compare degrees of two different parabolic subschemes of a reductive group scheme (see [Be2]).

Then, for a semistable G -bundle E , we bound the degree of the canonical reduction of any Frobenius pull back $(F^k)^*E$ (see Corollary 6.6). The proof is similar to the proof of [La1], Corollary 6.2. This result can be used to bound the degree of the associated bundle $E(\mathfrak{g})$ of Lie algebras.

This can be used to reduce the problem of boundedness of the family of semistable principal G -bundles with fixed numerical data to the boundedness of torsion free sheaves. Then as a corollary of [La1], Theorem 4.2, we get the following theorem (see Theorem 7.4):

Theorem 0.1. *Let G be a connected reductive group over k . Let us fix a polynomial P and some constant C . Then the family of all (rational) principal G -bundles E on X such that the degree of the canonical parabolic of E is $\leq C$, the degree of E is fixed, and the Hilbert polynomial of a torsion free sheaf extending $E(\mathfrak{g})$ is equal to P , is bounded.*

In particular, the above theorem implies that the family of semistable principal (rational) G -bundles with fixed degree and the Hilbert polynomial is bounded. In the case when X is a curve over an algebraically closed field of characteristic zero, the boundedness of the family of semistable principal G -bundles was proved by A. Ramanathan (see [Ra2] and [Ra3]). His method works also in the higher dimensional case.

In the case when X is a curve over an algebraically closed field of positive characteristic, the boundedness was first proved by K. Behrend in his thesis (see [Be1], Theorem 8.2.6), using Harder's result on reductions of G -bundles to a Borel subgroup of G . The same proof was later published by Y. Holla and M. S. Narasimhan (see [HN]).

Once we have Corollary 6.6 our method is quite similar to Ramanathan's method (with additional complications caused by higher dimension and positive characteristic).

Now assume for simplicity that G is semisimple and let $\rho: G \rightarrow \mathrm{SL}(V)$ be a homomorphism. Then the above results also allow us to bound the slope of the maximal destabilizing subsheaf of $E(V)$ for a semistable G -bundle E (see Theorem 8.4). In particular, if the characteristic of the field is large and E is semistable then we prove that $E(V)$ is also semistable.

A similar theorem, but with usually better bounds on the characteristic, was proved by S. Ilangoan, V. B. Mehta and A. J. Parameswaran (see [IMP]). Our approach has the advantage of giving some information for small primes.

In the forthcoming paper we will show how to apply the above results and methods of [La1] to obtain restriction theorems for principal G -bundles (see [BG] for some restriction theorems in the characteristic zero case).

There is a recent preprint by F. Coiai and Y. Holla (see [math.AG/0312280](#)) in which the authors prove a non-effective version of our Theorem 8.4 in the case when $H = \mathrm{GL}(V)$. They do it refining the methods of [RR] and, to the author, it does not seem possible to obtain effective bounds on instability of associated bundles using their method. They also try to use the above result to prove a weak version of boundedness for semistable principal G -bundles defined over the whole variety.

The structure of the paper is as follows. In Section 1 we recall a few results needed in the following. In Section 2 we define and study the canonical reduction. In Section 3 we define the strong canonical reduction and we study its properties. In Section 4 we explain a geometric meaning of complementary polyhedra introduced by Behrend in [Be2]. In Section 5 we prove that some Frobenius pull back of a principal G -bundle admits the strong canonical reduction. In Section 6 we study differences between semistability of a principal G -bundle and of its Frobenius pull back. We apply these results in Section 7 to get the boundedness of semistable principal G -bundles. In Section 8 we show how to bound the degree of instability of extensions of semistable G -bundles.

Notation.

We fix some notation used throughout the paper. Let k be an algebraically closed field of arbitrary characteristic. Let X be a normal projective geometrically connected variety defined over k . An open subset U of X is called *big*, if the complement of U has codimension ≥ 2 . A *rational vector bundle* E is a vector bundle defined over some big open subset of X . In this case E (or, more precisely, the associated locally free sheaf) has a unique extension \tilde{E} to a reflexive sheaf on X .

Let d be the dimension of X and let H_1, \dots, H_{d-1} be ample divisors on X . Then we can define the degree of E as the degree of its extension \tilde{E} with respect to $H_1 \dots H_{d-1}$. Using it one can easily define slopes and stability of rational vector bundles (cf. [La2], Appendix). In this paper, unless otherwise stated, we will always talk about semistability defined with respect to the above 1-cycle.

Let G be a connected reductive group over k . Then we define a *rational G -bundle* as a principal G -bundle on a big open subset of the smooth part of X . In this paper when writing “a principal G -bundle” we will always mean only a rational G -bundle.

Let $R(G)$ denote the radical of G . Since G is reductive, $R(G)$ is equal to the identity component of the reduced centre of G .

1. PRELIMINARIES

1.1. Here we recall some basic facts about parabolic subgroups in reductive groups.

Let G be a connected reductive algebraic group over k and let \mathfrak{g} denote its Lie algebra. Let us fix a maximal torus T in G and a Borel subgroup B containing it. Let $X^*(T)$ be the character group of T and let $\Phi = \Phi(G, T) \subset X^*(T)$ be the set of roots of G with respect to T . By definition Φ is the set of non-zero weights of T in \mathfrak{g} , acting via the adjoint representation Ad . The choice of B determines the set Φ^+ of positive roots which contains the subset Δ consisting of simple roots.

For any root $\alpha \in \Phi$ there exists an isomorphism x_α of G_α onto a unique closed subgroup X_α of G such that for any $t \in T$ and $a \in G_\alpha$ we have $tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$.

There is a 1 – 1 correspondence between subsets I of Δ and parabolic subgroups P_I containing B . There are two possible choices to define this correspondence. We do it in such a way that the Levi subgroup L_I of P_I containing T is generated by T and $X_{\pm\alpha}$ for $\alpha \in \Delta - I$. Then B corresponds to Δ and G corresponds to \emptyset .

Let us fix I and let Φ_I be the subset of Φ^+ consisting of those roots that are linear combinations of roots in $\Delta - I$. Any root $\alpha \in \Phi^+ - \Phi_I$ can be written as $\alpha = \sum_{\alpha_i \in \Delta} n_i \alpha_i$, where $n_i \geq 0$ and $l(\alpha) = \sum_{\alpha_i \in I} n_i > 0$. The number $l(\alpha)$ is called the *level* of α and $S(\alpha) = \sum_{\alpha_i \in I} n_i \alpha_i$ is called the *shape* of α .

For each non-zero shape S we set $V_S = \prod_{S(\alpha)=S} X_\alpha$. Each V_S is a module over the Levi subgroup $L = L_I$ of $P = P_I$ acting by inner automorphisms. One can also see that $R(L)$ acts on V_S by scalars.

The unipotent radical $R_u(P)$ is generated by X_α for $\alpha \in \Phi^+ - \Phi_I$ and it has a natural filtration $U_m \subset \cdots \subset U_1 \subset U_0 = R_u(P)$ such that $U_i \triangleleft R_u(P)$. U_i is defined as $\prod_{l(\alpha) > i} X_\alpha$. For each factor of this filtration we have the decomposition $U_i/U_{i+1} = \bigoplus_{l(S)=i+1} V_S$ into a direct sum of L -modules. If G is not special then each V_S is a simple L -module and the filtration is the socle (Loewy) series of $R_u(P_I)$, where $R_u(P)$ is treated as a P -module with P acting by inner automorphisms (see [ABS], Lemma 4). However, if G is special it can happen that V_S is not a simple L -module (see [ABS], Section 3, Remark 1).

Let \mathfrak{g}_α be the Lie algebra of X_α . Then we have an induced filtration of the Lie algebra \mathfrak{u} of $R_u(P)$ with quotients being L -modules, and we can identify the corresponding L -module $\bigoplus_{S(\alpha)=S} \mathfrak{g}_\alpha$ with V_S .

Then one can see that $\mathfrak{g}/\mathfrak{p}$, where \mathfrak{p} is the Lie algebra of P , has a dual filtration $W_m \subset \cdots \subset W_1 \subset W_0 = \mathfrak{g}/\mathfrak{p}$ such that $W_i/W_{i+1} = \bigoplus_{l(S)=i+1} V_S^* = \bigoplus_{l(S)=-(i+1)} V_S$ (cf. [ABS], Section 3, Remark 6).

1.2. We also need to recall some notation from [La1]. If E is a rational vector bundle on X defined over a field of characteristic p then we set

$$L_{\max}(E) = \lim_{k \rightarrow \infty} \frac{\mu_{\max}((F^k)^* E)}{p^k}.$$

This is a well defined rational number (this follows from [La1], Theorem 2.7). Similarly, one can also define $L_{\min}(E)$.

Note that in the introduction we proved that

$$L_{\max}(E_1 \otimes E_2) = L_{\max}(E_1) + L_{\max}(E_2)$$

for any two rational vector bundles E_1 and E_2 .

Theorem 1.3. ([La1], Corollary 6.2) *Let E be a rational vector bundle of rank r .*

- (1) *If $\mu_{\max}(\Omega_X) \leq 0$ then $L_{\max}(E) = \mu_{\max}(E)$ and $L_{\min}(E) = \mu_{\min}(E)$.*
- (2) *If $\mu_{\max}(\Omega_X) > 0$ then*

$$L_{\max}(E) \leq \mu_{\max}(E) + \frac{r-1}{p} L_{\max}(\Omega_X)$$

and

$$L_{\min}(E) \geq \mu_{\min}(E) - \frac{r-1}{p} L_{\max}(\Omega_X).$$

In Sections 6 and 8 we will prove similar theorems for principal G -bundles.

1.4. Let X be a d -dimensional normal variety defined over an algebraically closed field k and let H be an ample divisor on X . Let E be a rank r torsion free sheaf on X . Then there exist integers $a_0(E), \dots, a_d(E)$ such that

$$\chi(X, E(mH)) = \sum_{i=1}^d a_i(E) \binom{m+d-i}{d-i}.$$

Theorem 1.5. ([La1], Theorem 4.4) *Let μ_{\max}, a_0, a_1 and a_2 be some fixed numbers. Then the family of torsion free sheaves on X such that $\mu_{\max}(E) \leq \mu_{\max}$, $a_0(E) = a_0$, $a_1(E) = a_1$ and $a_2(E) \geq a_2$ is bounded, i.e., there exists a scheme S of finite type over k and an S -flat sheaf \mathcal{E} on $X \times S$ such that each member of the above family is contained in $\{\mathcal{E}_s\}_{s \in S}$, where \mathcal{E}_s is the restriction of \mathcal{E} to the fibre of the canonical projection over $s \in S$.*

Since $a_0(E) = rH^d$ and $a_1(E) = (c_1 E - \frac{r}{2} K_X) H^{d-1}$, fixing $a_0(E)$ and $a_1(E)$ is equivalent to fixing the rank r of E and the degree $c_1(E) H^{n-1}$ of E .

In the case X is smooth we can define the discriminant $\Delta(E)$ of E as $2rc_2 - (r-1)c_1^2$. It is easy to see that the condition $a_2(E) \geq a_2$ is equivalent to the condition $\Delta(E) H^{d-2} \leq C_X(r, a_1, a_2)$ for some explicit function C_X depending only on X and H (it is also equivalent to bounding $c_2(E) H^{d-2}$ from the above).

2. HARDER–NARASIMHAN FILTRATION

2.1. Let G be a connected reductive group over k and let E be a rational G -bundle on X . Let $E(G) = E \times_{G, \text{Int}} G$ denote the group scheme associated to E by the action of G on itself by inner automorphisms. Then we define the degree of $E(G)$ as the degree of the Lie algebra bundle $E(\mathfrak{g}) = E \times_{G, \text{Ad}} \mathfrak{g}$ of $E(G)$ on X considered as a rational vector bundle on X .

Let us fix a maximal torus T in G and some Borel subgroup B containing T . Let P be a parabolic subgroup of G containing B and let E_P be a (rational) reduction of its structure group to a parabolic subgroup P . Since every parabolic subgroup of G is conjugate to exactly one parabolic subgroup of G containing B , we do not restrict the class of considered reductions (cf. [Ra2], Remark 3.5.7).

Let $E_P(P) = E_P \times_{P, \text{Int}} P$ be the parabolic subgroup scheme of $E(G)$.

Definition 2.2.

The reduction E_P of E is called canonical (or the Harder–Narasimhan filtration) if it satisfies the following conditions:

- (1) for any parabolic subgroup scheme $Q \subset E(G)$ we have $\deg Q \leq \deg E_P(P)$,
- (2) $E_P(P)$ is maximal among all parabolic subgroup schemes \mathcal{P} of $E(G)$ that satisfy (1), i.e., if \mathcal{P} satisfies (1) and contains $E_P(P)$, then $\mathcal{P} = E_P(P)$.

The degree $\deg E_P(P)$ of the canonical reduction is denoted by $\deg_{\text{HN}} E$. This is a well defined integer. This follows from the fact that if E_P is a reduction of the structure group of E to P then $\deg E_P(P) \leq \text{rk } E(\mathfrak{g}) \cdot \mu_{\max}(E(\mathfrak{g}))$, since $E_P(\mathfrak{p}) \subset E(\mathfrak{g})$ (this proof works in general; cf. [Be2], Lemma 4.3 for the curve case). Note that $\deg_{\text{HN}} E \geq 0$, since G is also parabolic and $\deg E(G) = 0$.

Definition 2.3.

E is called slope semistable if and only if $\deg_{\text{HN}} E = 0$, i.e., if the degree of any parabolic subgroup scheme of $E(G)$ is non-positive.

*E is called strongly slope semistable if and only if $\text{char } k = 0$ or $\text{char } k > 0$ and $(F^l)^*E$ is slope semistable for all $l \geq 0$, where F denotes the Frobenius morphism.*

The above definitions are not completely standard if one tries to understand them in the vector bundle case. In this case one can easily interpret the definition in the following way. To any sheaf G we can associate the point $p(G) = (\text{rk } G, \deg G)$ in the plane. Let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be a filtration of a vector bundle E with torsion free quotients. We can successively connect the points $p(E_0), \dots, p(E_m)$ by line segments. If we also connect $p(E_m)$ with $p(E_0)$ then we obtain a generalized polygon. One can easily see that the area of this polygon is equal to half of the degree of the corresponding parabolic subscheme of the associated group scheme.

The polygon corresponding to the Harder–Narasimhan filtration is called the *Harder–Narasimhan polygon* and it is denoted by $\text{HNP}(E)$. Now Definition 2.2 says that the Harder–Narasimhan filtration corresponds to the polygon with the largest area. This is clear, since the Harder–Narasimhan polygon lies over all the polygons obtained from the filtrations of E . This and a small computation imply the following proposition:

Proposition 2.4. *Let E be a rational $\text{GL}(V)$ -bundle and let $E(V)$ be the corresponding rational vector bundle. Let r_i, μ_i denote ranks and slopes of the quotients of the Harder–Narasimhan filtration of $E(V)$. Let μ_{\max}, μ_{\min} denote the corresponding slopes for $E(V)$. Then*

$$\deg_{\text{HN}} E = 2 \text{ area } \text{HNP}(E(V)) = \sum_{i < j} r_i r_j (\mu_i - \mu_j).$$

In particular, we have

$$(r - 1)(\mu_{\max} - \mu_{\min}) \leq \deg_{\text{HN}}(E) \leq \frac{r^2}{4}(\mu_{\max} - \mu_{\min}),$$

where $r = \dim V$.

If F is a vector bundle then $\deg_{\text{HN}} F$ will denote $2 \text{ area } \text{HNP}(F)$, which by the above proposition is equal to the degree of the canonical parabolic of the corresponding principal bundle.

If E is a principal G -bundle and k is a field of characteristic p then we set

$$\deg_{\text{HN}, l} E = \frac{\deg_{\text{HN}}(F^l)^* E}{p^l}.$$

Lemma 2.5. *The sequence $\{\deg_{\text{HN},l} E\}$ is non-decreasing and it has a finite limit denoted by $\deg_{\text{HN},\infty} E$.*

Proof. The first part follows immediately from the definition of the canonical reduction. To prove existence of the limit it is sufficient to note that if \tilde{E}_P is the canonical reduction of $(F^k)^*E$ then the vector bundle $\tilde{E}_P(\mathfrak{p})$ is contained in the vector bundle $(F^k)^*E(\mathfrak{g})$, so its degree is less or equal to $p^k \text{rk } E(\mathfrak{g}) \cdot L_{\max}(E(\mathfrak{g}))$ (see [La], Corollary 2.5). In particular, $\deg_{\text{HN},\infty} E \leq \text{rk } E(\mathfrak{g}) \cdot L_{\max}(E(\mathfrak{g}))$, Q.E.D.

Let L be a Levi subgroup of P . We have a natural projection $P \rightarrow P/R_u(P)$, where $R_u(P)$ is the unipotent radical of P . By the definition of a Levi subgroup we have $P/R_u(P) \simeq L$, so each principal P -bundle has an extension to an L -bundle.

Let us also recall that the unipotent radical $R_u(P)$ has the filtration $U_m \subset U_{m-1} \subset \cdots \subset U_1 = R_u(P)$ in which the quotients U_i/U_{i+1} are direct sums of simple L -modules V_S for all shapes of level $i+1$ (see 1.1). Each V_S is also a k -vector space.

Theorem 2.6. ([Be2], Theorem 7.3) *Every principal G -bundle has a canonical reduction E_P to some parabolic subgroup P of G . This reduction satisfies the following conditions:*

- (1) *the extension E_L of E_P to L is a semistable rational L -bundle,*
- (2) *for all shapes S of positive level the associated vector bundle $E_L(V_S)$ has positive degree.*

Moreover, if (1) and (2) are satisfied for some reduction E_P of E to some parabolic subgroup P of G , then this reduction is canonical.

The above theorem is formulated in [Be2] only for principal G -bundles (or, more generally, group schemes) over a curve. However, its proof with minor modifications works in general. One should only note that $\deg_{\text{HN}} E$ is well defined and then repeatedly use the fact that any generic section of a parabolic subscheme of $E(G)$ extends to some big open subset of X (cf. [RR], Section 4).

In [Be2] condition (2) is asserted only for shapes of level 1, but it is true for all shapes of positive level (see Lemma 4.5).

In the vector bundle case (1) of Theorem 2.6 corresponds to convexity of the Harder–Narasimhan polygon and (2) corresponds to semistability of quotients in the Harder–Narasimhan filtration.

The following theorem was proved by Ramanan and Ramanathan:

Theorem 2.7. ([RR], Theorem 3.23; see also [La2], Theorem A.3) *Let $\rho: G \rightarrow H$ be a homomorphism of connected reductive k -groups and assume that $\rho(R(G)) \subset R(H)$. Let E_G be a rational G -bundle and let E_H be the rational H -bundle obtained from E by extension. If E_G is strongly semistable then E_H is also strongly semistable.*

Corollary 2.8. *Let E be a principal G -bundle. If $E(\mathfrak{g})$ is semistable as a vector bundle then E is semistable. In particular, E is strongly semistable if and only if $E(\mathfrak{g})$ is strongly semistable.*

Proof. If E is not semistable then $E(\mathfrak{g})$ is a degree zero vector bundle which contains the vector bundle $E_P(\mathfrak{p})$ of degree > 0 . Hence $E(\mathfrak{g})$ is not semistable.

If E is strongly semistable then $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ maps the radical of G to the identity, so by Theorem 2.7 $E_{\text{GL}(\mathfrak{g})}$ is a strongly semistable principal $\text{GL}(\mathfrak{g})$ -bundle. This implies that the associated vector bundle is strongly semistable. Q.E.D.

Let us recall that canonical reduction is functorial under separable base change:

Lemma 2.9. (see [Be2], Corollary 7.4) *Let $\pi: Y \rightarrow X$ be a finite separable morphism of normal projective varieties over k . Let E_P be the canonical reduction of a rational G -bundle E defined over X . Then π^*E_P is the canonical reduction of π^*E .*

The following proposition is an analogue of the Ramanan–Ramanathan theorem but we do not need to assume strong semistability. It immediately implies Theorem 2.7 in the case ρ is surjective.

Proposition 2.10. *Let $\rho: G \rightarrow H$ be a surjective homomorphism of connected reductive groups. Assume that the kernel group scheme of ρ is contained in the centre group scheme $Z(G)$ of G . Let E be a principal G -bundle with the Harder–Narasimhan filtration E_P . Then the extension of structure group of E_P to the image Q of P is the Harder–Narasimhan filtration of the extension E_H of structure group of E to H .*

Proof. Since $\ker \rho \subset Z(G) \subset P$, we have $G/P \simeq (G/\ker \rho)/(P/\ker \rho) = H/Q$. Since $\mathfrak{g}/\mathfrak{p}$ is the tangent space at e to G/P , we have $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{h}/\mathfrak{q}$ and hence

$$\deg E_Q(\mathfrak{q}) = -\deg E_Q(\mathfrak{h}/\mathfrak{q}) = -\deg E_P(\mathfrak{g}/\mathfrak{p}) = \deg_{\text{HN}} E.$$

If $E_{Q'}$ is a reduction of E_H to a parabolic subgroup $Q' \subset H$ and $P' = \rho^{-1}(Q')$, then an isomorphism $G/P' \simeq H/Q'$ induces a reduction $E_{P'}$ of E to the parabolic P' (let us recall that such a reduction can be treated as a rational section of $E(G/P') \rightarrow X$). By a similar computation as above we have

$$\deg E_{Q'}(\mathfrak{q}') = \deg E_{P'}(\mathfrak{p}') \leq \deg_{\text{HN}} E = \deg E_Q(\mathfrak{q}).$$

This shows that E_Q satisfies condition (1) of Definition 2.2. Similarly one can check that E_Q satisfies condition (2), Q.E.D.

As a corollary we see that in arbitrary characteristic a principal G -bundle is semistable if and only if its extension to the adjoint form $\text{Ad } G$ is semistable (cf. Corollary 2.8). In the following we do not use this fact.

3. STRONG HARDER–NARASIMHAN FILTRATION

Definition 3.1. *Let E be a principal G -bundle. We say that E_P is the strong Harder–Narasimhan filtration of E (or that E_P is the strong canonical reduction of E) if it satisfies condition (2) of Theorem 2.6 and the extension E_L is strongly semistable.*

Obviously, if k has positive characteristic then not every principal G -bundle has a strong Harder–Narasimhan filtration. However, on some special manifolds (e.g., with a globally generated tangent bundle) every semistable G -bundle is strongly semistable and then every principal G -bundle has a strong Harder–Narasimhan filtration (see Corollary 6.4).

In [Be2], Conjecture 7.6, Behrend conjectures that if E_P is the canonical reduction of E then

This conjecture was proved by V. B. Mehta and S. Subramanian under the assumption that the characteristic of the base field is large (see [MS], Corollary 3.6).

Here we prove that this result holds in an arbitrary characteristic for some Frobenius pull back of E (see Theorem 5.1):

Proposition 3.2. *Assume that a principal G -bundle E has the strong Harder–Narasimhan filtration E_P . Then*

$$h^0(X, E_P(\mathfrak{g}/\mathfrak{p})) = 0.$$

Proof. Let us note that $E_P(\mathfrak{g}/\mathfrak{p})$ has a filtration by vector bundles in which the quotients are duals of vector bundles $E_L(V_S)$ for shapes of level ≥ 1 . Since the radical $R(L)$ acts on V_S by scalars and E_L is strongly semistable, it follows that $E_L(V_S)$ is also strongly semistable (see Theorem 2.7). Since each $E_L(V_S)$ has positive degree, we have $\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0$ and in particular $E_P(\mathfrak{g}/\mathfrak{p})$ has no sections, Q.E.D.

Proposition 3.3. *Assume that E has strong Harder–Narasimhan filtration E_P . Let*

$$E_{-r} \subset \cdots \subset E_{-1} \subset E_0 \subset \cdots \subset E_s = E(\mathfrak{g})$$

be the Harder–Narasimhan filtration of $E(\mathfrak{g})$ indexed in such a way that $\mu(E_i/E_{i-1}) < 0$ for $i \geq 1$ and $\mu(E_i/E_{i-1}) \geq 0$ for $i \leq 0$. Then $E_0 = E_P(\mathfrak{p})$ and $E_{-1} = E_P(\mathfrak{u})$. In particular, $E_0/E_{-1} = E_L(\mathfrak{l})$ is strongly semistable of degree 0, $r = s$ and $E(\mathfrak{g})/E_0$ is isomorphic to E_{-1}^ as a rational vector bundle.*

Proof. Let us recall that $\mathfrak{u} = \text{Lie } R_u(P)$ is filtered with L -modules V_S , where S are shapes of level ≥ 1 . Then $\mathfrak{g}/\mathfrak{p}$ is filtered with dual L -modules V_S^* . Hence $E(\mathfrak{g}/\mathfrak{p}) = E(\mathfrak{g})/E_P(\mathfrak{p})$ has a filtration, whose quotients are strongly semistable vector bundles $E_L(V_S)^*$ of negative degree. In particular, $\mu_{\max}(E(\mathfrak{g})/E_P(\mathfrak{p})) < 0$. Since $\mu_{\min}(E_0) \geq 0$, this implies that the map $E_0 \rightarrow E(\mathfrak{g})/E_P(\mathfrak{p})$ is zero, i.e., $E_0 \subset E_P(\mathfrak{p})$.

Note that $E_P(\mathfrak{p})/E_P(\mathfrak{u}) = E_L(\mathfrak{l})$ is strongly semistable and has degree 0 (as a bundle of reductive Lie algebras). Moreover, $E_P(\mathfrak{u})$ has a filtration with quotients that are strongly semistable sheaves $E_P(V_S)$ of positive degree. Hence $\mu_{\min}(E_P(\mathfrak{p})) = 0 < \mu_{\max}(E(\mathfrak{g})/E_0)$, which implies that the map $E_P(\mathfrak{p}) \rightarrow E(\mathfrak{g})/E_0$ is zero, i.e., $E_P(\mathfrak{p}) = E_0$.

Now the map $E_{-1} \rightarrow E(\mathfrak{g})/E_P(\mathfrak{u})$ is zero since $\mu_{\min}(E_{-1}) = \mu(E_{-1}/E_{-2}) > \mu(E_0/E_{-1}) \geq 0 = \mu_{\max}(E(\mathfrak{g})/E(\mathfrak{u}))$. Note that $\mu_{\max}(E(\mathfrak{g})/E_{-1}) = \mu(E_0/E_{-1}) \leq 0$. This follows from the fact that $E_0 = E_P(\mathfrak{p})$ has a filtration with strongly semistable quotients, whose slopes are non-positive. Since $\mu_{\min}(E_P(\mathfrak{u})) > 0$ it follows that the map $E_P(\mathfrak{u}) \rightarrow E(\mathfrak{g})/E_{-1}$ is zero. Hence $E_{-1} = E_P(\mathfrak{u})$, Q.E.D.

The above proposition should be compared with the construction in [AB].

Corollary 3.4. *Let E be a principal G -bundle which has strong Harder–Narasimhan filtration E_P . Let $E(\mathfrak{g})$ be the vector bundle associated to E by the adjoint representation $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$. Then*

$$(\dim \mathfrak{g} + \dim \mathfrak{l}) \cdot \deg_{\text{HN}} E \leq \deg_{\text{HN}} E(\mathfrak{g}) \leq 2 \dim \mathfrak{g} \cdot \deg_{\text{HN}} E,$$

where L is the Lie algebra of a Levi subgroup of P .

Proof. Draw the Harder–Narasimhan polygon $A = \text{HNP}(E(\mathfrak{g}))$. By Proposition 3.3, A is contained in the rectangle whose two vertices are equal to 0 and $p(E(\mathfrak{g}))$ and one side contains points $p(E_{-1})$ and $p(E_0)$. Thus Proposition 2.4 implies that

$$\deg_{\text{HN}} E(\mathfrak{g}) = 2 \text{ area}(A) \leq 2 \dim \mathfrak{g} \cdot \deg_{\text{HN}} E.$$

Another inequality follows from the fact that A contains the trapezium with vertices $0, p(E(\mathfrak{g})), p(E_0)$ and $p(E_{-1})$, Q.E.D.

Corollary 3.5. *Let G and H be connected reductive groups and let $\varphi: G \rightarrow H$ be a homomorphism. Let E_G be a principal G -bundle and E_H be the principal H -bundle obtained from E by extension. Let $\phi: E_G(\mathfrak{g}) \rightarrow E_H(\mathfrak{h})$ be the induced homomorphism of Lie algebra bundles. Assume that both E_G and E_H have strong Harder–Narasimhan filtrations E_P and E_Q , respectively. Then $\phi(E_P(\mathfrak{p})) \subset E_Q(\mathfrak{q})$.*

Proof. By Proposition 3.3 we have $\mu_{\min}(E_P(\mathfrak{p})) \geq 0 > \mu_{\max}(E_H(\mathfrak{h})/E_Q(\mathfrak{q}))$. Hence the map $E_P(\mathfrak{p}) \rightarrow E_H(\mathfrak{h})/E_Q(\mathfrak{q})$ is zero, Q.E.D.

Remark. If k is a field of characteristic zero then Atiyah and Bott showed that the canonical reduction is functorial with respect to φ (see [AB], Proposition 10.4). One cannot hope that this is true in the positive characteristic case. However, from the result of Ilangoan, Mehta and Parameswaran (see [IMP]) one can see that the canonical reduction is functorial if the characteristic of the field is large enough (weaker bounds can be obtained from Theorem 8.4). One need only to take such $p = \text{char } k$ that the kernel and the cokernel of $\mathfrak{g} \rightarrow \mathfrak{h}$ are, as G -modules, the direct summands of \mathfrak{g} and \mathfrak{h} , and use Proposition 3.3 (or better [MS], Proposition 2.2).

4. COMPLEMENTARY POLYHEDRA AND ELEMENTARY VECTOR BUNDLES

In this section we introduce complementary polyhedra and we prove some auxiliary results about elementary vector bundles. The best place to find the necessary definitions and basic properties is Behrend’s paper [Be2] or his PhD thesis [Be1].

4.1. Let the notation be as in 1.1 and 2.1. Let V be a real vector subspace of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the set $\Phi = \Phi(G, T)$ of roots of G with respect to T . Then (V, Φ) forms a root system. For any $\alpha \in \Phi$ let α^\vee denote the corresponding coroot in V^* . Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots corresponding to the choice of B and let $\lambda_1, \dots, \lambda_n$ be the dual basis of $\alpha_1^\vee, \dots, \alpha_n^\vee$. This basis forms the set of fundamental dominant weights with respect to B and $\lambda_1, \dots, \lambda_n$ are vertices of the Weyl chamber \mathfrak{c} corresponding to the choice of B .

Let E be a principal G -bundle on X and let E_P be a reduction of structure group of E to some parabolic subgroup P of G containing B . The parabolic P corresponds to some subset $\{\alpha_i\}_{i \in I}$ of Δ (see 1.1), where $I \subset \{1, \dots, n\}$. The facet of \mathfrak{c} with vertices $\{\lambda_i\}_{i \in I}$ corresponds to P and it will be denoted by the same letter.

Let K denote the function field of X . Take a maximal torus $T_K \subset E(G)_K$, which is contained in $E_P(P)_K$. If T_K is not split then let us pass to a separable cover $\pi: Y \rightarrow X$, where $T_{K'} = \pi^* T_K$ splits (K' is the function field of Y). Let $B_{K'}$ be the Borel subgroup contained in $\pi^* E_P(P)_{K'}$. We can choose an isomorphism of root systems $\Psi = \Phi(\pi^* E(G)_{K'}, T_{K'})$ and Φ such that the Weyl chambers corresponding to $B_{K'}$ and B become equal and the facet corresponding to the parabolic

$\pi^*E_P(P)_{K'}$ corresponds to P . Let us set

$$d(\mathfrak{c}) = \sum_{i=1}^n \deg B_{K'}(V_{S(\alpha_i)}) \cdot \lambda_i^\vee,$$

where $V_{S(\alpha_i)}$ are considered with respect to B . The vector $d(\mathfrak{c})$ is well defined since $B_{K'}$ uniquely extends to the Borel subgroup scheme of $\pi^*E(G)$. It corresponds to the vector $d(\mathfrak{c}_{K'})$ that belongs to the complementary polyhedron for Ψ (see [Be2], Proposition 6.6).

4.2. Now let us consider L -modules V_S defined with respect to the Levi subgroup L of P (see 1.1). All V_S corresponding to shapes S of level 1 are called *elementary L -modules*. The corresponding vector bundles $E_L(V_S)$ are called *elementary vector bundles* (see [Be2], Definition 5.5) associated to E_P . The degree of $E_L(V_S)$, where S is a shape of level 1 corresponding to a root α , is called the *numerical invariant* of E_P with respect to α and it is denoted by $n(E_P; \alpha)$. To be compatible with [Be2] we need to define numerical invariants using fundamental weights. Namely, let λ be the fundamental weight of G with respect to B dual to the coroot α^\vee . Then we set $n(E_P; \lambda) = n(E_P; \alpha)$, defining numerical invariants with respect to fundamental weights corresponding to P .

One can see that the degree of $E_L(V_S)$ for a shape S of positive level is a non-negative linear combination of numerical invariants of E_P with respect to simple roots $\alpha \in I$, where I is the subset of the set of simple roots corresponding to P (see Lemma 4.5).

Lemma 4.3. *Let $\pi: Y \rightarrow X$ be a finite morphism of normal projective varieties over k . Let E_P be the canonical reduction of a rational G -bundle E defined over X . Then the numerical invariants of π^*E_P are non-negative.*

Proof. In case π is separable the lemma follows from [Be2], Lemma 7.1. So we can assume that π is equal to the Frobenius morphism $F: X \rightarrow X$. In this case the lemma follows immediately from the fact that $(F^*E_P)(V_S) = F^*(E_P(V_S))$ and the degree of a rational vector bundle multiplies by $p = \text{char } k$ under the Frobenius morphism, Q.E.D.

4.4. Now let us set

$$U(P) = \{\alpha \in \Phi: \text{there exists } \lambda \in \text{vert } P \text{ such that } \langle \alpha, \lambda^\vee \rangle > 0\}.$$

This set is equal to $\Phi^+ - \Phi_I$ and it has a natural decomposition into subsets corresponding to roots of the same level (with respect to P). In the above notation $U(P)$ decomposes into subsets

$$\Psi(P, \sum_{i \in I} n_i \lambda_i) = \{\alpha \in \Phi: \langle \alpha, \lambda_i^\vee \rangle = n_i \text{ for every } i \in I\}.$$

Note that $\Psi(P, \sum_{i \in I} n_i \lambda_i)$ is the set of all roots of shape $S = \sum_{i \in I} n_i \alpha_i$. Moreover, we have

$$\text{rk } E_P(V_S) = |\Psi(P, \sum_{i \in I} n_i \lambda_i)|$$

and

$$\deg E_P(V_S) = \sum_{\alpha \in \Psi(P, \sum_{i \in I} n_i \lambda_i)} \langle \alpha, d(\mathfrak{c}) \rangle.$$

Lemma 4.5. *Let $\alpha = \sum_{i=1}^n n_i \alpha_i \in U(P)$. Then*

$$\mu(E_P(V_{S(\alpha)})) = \sum_{i \in I} n_i \frac{n(P, \lambda_i)}{|\Psi(P, \lambda_i)|} = \sum_{i \in I} n_i \mu(E_P(V_{S(\alpha_i)})).$$

Proof. Let us set

$$y(P) = \sum_{\lambda \in \text{vert } P} \frac{n(P, \lambda)}{|\Psi(P, \lambda)|} \cdot \lambda^\vee.$$

Then for all vertices of P we have $\langle \lambda, d(\mathfrak{c}) \rangle = \langle \lambda, y(P) \rangle$. Now let us note that $\sum_{\alpha \in \Psi(P, \sum_{i \in I} n_i \lambda_i)} \alpha$ belongs to the vector space spanned by $\{\lambda_i\}_{i \in I}$ (cf. [Be2], Lemma 3.6). Hence

$$\deg E_P(V_{S(\alpha)}) = \sum_{\alpha \in \Psi(P, \sum_{i \in I} n_i \lambda_i)} \langle \alpha, y(P) \rangle = |\Psi(P, \sum_{i \in I} n_i \lambda_i)| \cdot \sum_{i \in I} n_i \frac{n(P, \lambda_i)}{|\Psi(P, \lambda_i)|},$$

Q.E.D.

5. ASYMPTOTIC STRONG HARDER–NARASIMHAN FILTRATION

In the vector bundle case we proved that some Frobenius pull back has strong Harder–Narasimhan filtration (see [Lal], Theorem 2.7). We prove that the same result holds for principal G -bundles.

Theorem 5.1. *Let E be a principal G -bundle on X defined over a field k of positive characteristic p and let $F: X \rightarrow X$ be the Frobenius morphism. Then there exists l such that $(F^l)^* E$ has strong Harder–Narasimhan filtration.*

Proof. To each rational G -bundle E_P one can associate a sequence of parabolic subgroups $B \subset P_k$ corresponding to canonical reductions E_{k, P_k} of $(F^k)^* E$. Since elements of this sequence are chosen from a finite set of parabolic subgroups that contain B , we can take a constant subsequence $\{P_{j_k}\}$ of $\{P_k\}$. Set $P = P_{j_k}$ and let $I = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$ be the corresponding subset determining P . Then we consider r sequences $\{n_{1,k}\}, \dots, \{n_{r,k}\}$ defined by

$$n_{i,k} = \frac{n(E_{j_k, P_{j_k}}; \alpha_i)}{p^{j_k}}$$

for $i = 1, \dots, r$ (see 4.2). By Lemma 4.5 there exist some nonnegative rational numbers A_1, \dots, A_r , depending only on the type of G and P , such that

$$\sum A_i n_{i,k} = \frac{\deg E_{j_k, P_{j_k}}}{p^{j_k}}.$$

Since the sequence $\left\{ \frac{\deg E_{j_k, P_{j_k}}}{p^{j_k}} \right\}$ converges to a finite number $\deg_{HN, \infty} E$ we can find a subsequence $\{j_{k_l}\}$ of $\{j_k\}$ such that all the sequences $\{n_{i, j_{k_l}}\}$ converge.

Then for any $\epsilon > 0$ we can find j_{k_l} and $j_{k_{l'}}$, where $l' > l$, such that

for $i = 1, \dots, r$. Let us replace E with $(F^{j_{k_l}})^*E$ and set $k = j_{k_{l'}} - j_{k_l}$. Let E_P be the Harder–Narasimhan filtration of E and assume that $(F^k)^*E_P$ is not the Harder–Narasimhan filtration of $(F^k)^*E$. Let \tilde{E}_P be the Harder–Narasimhan filtration of $(F^k)^*E$.

Let us set $\mathcal{G} = ((F^k)^*E)(G)$, $\mathcal{P} = \tilde{E}_P(P)$, $\mathcal{Q} = (F^k)^*(E_P(P))$ and let K denote the function field of X . Then there exists a maximal torus $T_K \subset \mathcal{G}_K$ which is contained in $\mathcal{P}_K \cap \mathcal{Q}_K$ (see [SGA3], exp. XXVI, 4.1.1). If T_K is not split then let us pass to a separable cover $Y \rightarrow X$, where T_K splits. By Lemma 2.9 we may actually replace X with Y and assume that T_K splits on X . Our assumption imply that $\mathcal{P}_K \neq \mathcal{Q}_K$. Let B_K and B'_K be the Borel subgroups contained in \mathcal{P}_K and \mathcal{Q}_K , respectively.

We can find such an isomorphism of root systems $\Phi(G, T)$ and $\Phi(\mathcal{G}_K, T_K)$ that the Weyl chambers \mathfrak{c} and \mathfrak{c}_K corresponding to B and B_K become equal and the facet P_K corresponding to \mathcal{P}_K corresponds to the facet P of \mathfrak{c} . We can also find such an isomorphism of root systems $\Phi(G, T)$ and $\Phi(\mathcal{G}_K, T_K)$ that the Weyl chambers \mathfrak{c} and \mathfrak{c}_K corresponding to B and B'_K become equal and the facet Q_K corresponding to \mathcal{Q}_K corresponds to the facet P of \mathfrak{c} . This shows that there exists an isomorphism σ of $V = X^*(T_K) \otimes_{\mathbb{Z}} \mathbb{R}$ preserving $\Phi = \Phi(\mathcal{G}_K, T_K)$ and such that the image of the facet P_K is equal to the facet Q_K . Obviously, σ is an element of the Weyl group W .

Let (\cdot, \cdot) be a scalar product on V that gives a W -invariant Euclidean metric on V . Let $\|\cdot\|$ be the associated norm. This can be used to identify V and V^* . In this identification if $\alpha \in \Phi$ then α^\vee is identified with $\frac{2\alpha}{(\alpha, \alpha)}$.

Let \mathfrak{C} be the set of Weyl chambers of Φ and let $d = (d(\mathfrak{c}))_{\mathfrak{c} \in \mathfrak{C}}$ consists of vectors in V^* that correspond to $\frac{1}{p^k}$ of vectors in the complementary polyhedron for Φ determined by $T_K \subset \mathcal{G}_K$ (see [Be2], Proposition 6.6). This set still forms a complementary polyhedron such that

$$n(P_K, \lambda) = \frac{n(\tilde{E}_P; \lambda)}{p^k} \geq 0$$

and

$$n(Q_K, \lambda) = \frac{n((F^k)^*E_P; \lambda)}{p^k} = n(E_P; \lambda) \geq 0$$

(cf. Lemma 4.3).

Our assumptions show that σ is an isometry such that $\sigma(P_K) = Q_K$ and

$$(*) \quad n(P_K; \lambda) \leq (1 + \epsilon) \cdot n(Q_K; \sigma(\lambda))$$

for all vertices λ of P_K . Let $\Psi(P, \lambda)$ be the elementary set of roots associated to facet P and its vertex λ (see 4.4). Obviously, $|\Psi(P_K; \lambda)| = |\Psi(Q_K; \sigma(\lambda))|$ for any vertex λ of P_K .

As in the proof of Lemma 4.5 let us set

$$y(P) = \sum_{\lambda \in \text{vert } P} \frac{n(P; \lambda)}{|\Psi(P, \lambda)|} \cdot \lambda^\vee$$

for a facet P . By assumption $y(P_K) \in P_K^\vee$ and $y(Q_K) \in Q_K^\vee$ (this follows from non-negativity of the numerical invariants of P_K and Q_K).

If we consider $y(Q_K)$ and $y(P_K)$ as vectors in V , then [Be2], Lemma 2.5 and [Be2], Proposition 3.13 show that

$$(\lambda, y(Q_K)) = \langle \lambda, d(\mathfrak{c}'_K) \rangle \leq (\lambda, y(P_K))$$

for any vertex λ of Q_K . This implies that

$$(**) \quad (y(Q_K), y(Q_K)) \leq (y(Q_K), y(P_K)).$$

Since $(y(Q_K), y(P_K)) = \|y(Q_K)\| \cdot \|y(P_K)\| \cdot \cos \alpha$, where α is the angle between $y(Q_K) \in Q_K^\vee$ and $y(P_K)^\vee$. Since $\mathcal{P} \neq \mathcal{Q}$, this angle is non-zero and $\cos \alpha$ cannot be larger than the maximum s_0 of $\cos \alpha$ over all non-zero angles between different facets in a partition of $(X^*(T) \otimes_{\mathbb{Z}} \mathbb{R})^*$ determined by $\Phi(G, T)^\vee$. Obviously, $s_0 < 1$, since $\Phi(G, T)$ is finite. Hence by $(**)$

$$\|y(Q_K)\| \leq s_0 \|y(P_K)\|.$$

But using $(*)$ we get

$$\begin{aligned} \|y(P_K)\|^2 &= \sum_{\lambda, \mu \in \text{vert } P_K} \frac{n(P_K; \lambda) \cdot n(P_K; \mu)}{|\Psi(P_K, \lambda)| \cdot |\Psi(P_K, \mu)|} \cdot (\lambda^\vee, \mu^\vee) \\ &\leq (1 + \epsilon)^2 \sum_{\lambda, \mu \in \text{vert } P_K} \frac{n(Q_K; \sigma(\lambda)) \cdot n(Q_K; \sigma(\mu))}{|\Psi(Q_K, \sigma(\lambda))| \cdot |\Psi(Q_K, \sigma(\mu))|} \cdot (\sigma(\lambda^\vee), \sigma(\mu^\vee)) \\ &= (1 + \epsilon)^2 \|x(Q_K)\|^2. \end{aligned}$$

Therefore $1 \leq s_0(1 + \epsilon)$ and for small ϵ we get a contradiction, Q.E.D.

Corollary 5.2. *Let E be a principal G -bundle on a curve C defined over a field k of positive characteristic p and let $F: C \rightarrow C$ be the Frobenius morphism. Then there exists l such that the Harder–Narasimhan filtration \tilde{E}_P of $(F^l)^*E$ has a strongly semistable reduction to the Levi component $L \subset P$.*

Proof. Let us take l such that $(F^l)^*E$ has the strong Harder–Narasimhan filtration. We can take even larger l such that the degrees of all elementary vector bundles are greater than $\deg K_C$. The latter can be easily achieved, since the degrees of elementary vector bundles are positive and they multiply by p under the Frobenius pull back.

The theorem will be proved if we show that the non-abelian cohomology group $H^1(C, E_P(R_u(P)))$ is trivial (cf. [SGA3], exp. XXVI, Corollaire 2.2 and [Su], Theorem 2.2).

Now note that $E_P(R_u(P))$ has a filtration whose quotients are vector bundles $E_P(V_S)$. Since the non-abelian cohomology of a vector bundle coincides with its usual sheaf cohomology, it is sufficient to prove that the sheaf cohomology $H^1(C, E_P(V_S))$ vanish. But by Serre duality $h^1(C, E_P(V_S)) = h^0(C, E_P(V_S)^* \otimes \omega_C)$ and $E_P(V_S)^* \otimes \omega_C$ is a semistable vector bundle of negative degree (semistability follows from Theorem 2.7), so it has no sections, Q.E.D.

The above corollary generalize [Ra1], Lemma 3.7 (where $C = \mathcal{P}^1$) and [Su], Theorem 2.2 (where C is an elliptic curve). In the vector bundle case it was first proved by V. B. Mehta and S. Subramanian and the author learned it from S. Subramanian. In this case the corollary says that if E is a vector bundle on a curve then the Harder–Narasimhan filtration of some Frobenius pull back of E splits into a direct

6. SEMISTABILITY OF FROBENIUS PULL BACKS

6.1. Let us fix a maximal torus T and a Borel subgroup $B \supset T$ in G . Let E be a principal G -bundle with canonical reduction E_P for some parabolic subgroup P containing B . Let λ be the fundamental weight corresponding to one of vertices of the facet corresponding to P . Let V_λ denote the elementary module $V_{S(\alpha)}$, where α is the simple root such that λ is dual to the coroot α^\vee .

By abuse of notation we will use the same notation to denote an elementary module corresponding to other parabolic subgroups of G . This will not lead into confusion since we use this notation together with a reduction of structure group of E to the parabolic with respect to which we consider it as an elementary module.

Let us recall that a parabolic subgroup of G is called *maximal*, if it is proper (i.e., different to G) and if it is not contained in any other proper parabolic subgroup of G . In the vector bundle case it is obvious that any component of the Harder–Narasimhan filtration destabilizes the vector bundle. The same fact holds for principal G -bundles:

Proposition 6.2. *Let $Q \subset G$ be a maximal parabolic subgroup containing P and let E_Q be the extension of structure group of E_P to Q . Let μ be the fundamental weight corresponding to Q . Then*

$$(6.2.1) \quad \mu(E_Q(V_\mu)) = \sum_{\lambda \in \text{vert } P} \frac{\langle \mu, \lambda^\vee \rangle}{\langle \mu, \mu^\vee \rangle} \mu(E_P(V_\lambda)),$$

where $\text{vert } P$ denote the set of fundamental weights corresponding to P . In particular, the degree of $E_Q(Q)$ is positive.

Proof. Let \mathfrak{c} be the Weyl chamber corresponding to B and let $d(\mathfrak{c})$ be as in 4.1. This vector will be used to compute the degree of $E_Q(Q)$. Let P' and Q' be the facets corresponding to $E_P(P)$ and $E_Q(Q)$, respectively. The facet Q' has only one vertex μ and it is also one of the vertices of P' .

Now let us note that

$$\langle \mu, d(\mathfrak{c}) \rangle = \langle \mu, y(P') \rangle = \sum_{\lambda \in \text{vert } P'} \frac{n(P', \lambda)}{|\Psi(P', \lambda)|} \langle \mu, \lambda^\vee \rangle = \sum_{\lambda \in \text{vert } P} \langle \mu, \lambda^\vee \rangle \cdot \mu(E_P(V_\lambda)),$$

since μ is a vertex of P' . But we also have

$$\langle \mu, d(\mathfrak{c}) \rangle = \langle \mu, y(Q') \rangle = \frac{n(Q', \mu)}{|\Psi(Q', \mu)|} \langle \mu, \mu^\vee \rangle = \langle \mu, \mu^\vee \rangle \cdot \mu(E_Q(V_\mu)),$$

since μ is also a vertex of Q' . Comparing these two equalities yields the required equality.

Now note that the degree of $E_Q(Q)$ is a positive multiple of $\mu(E_Q(V_\mu))$ (this follows, e.g., from Lemma 4.5). Since E_P is the canonical reduction we have $\mu(E_P(V_\lambda)) > 0$, so the inequality $\deg E_Q(Q) > 0$ follows from the fact that the coefficients in (6.2.1) are non-negative and one of them is equal to 1, Q.E.D.

The next theorem bounds the slopes of elementary vector bundles of Frobenius pull back of a principal G -bundle. The method of proof is similar to the proof of [La1], Corollary 6.2.

Theorem 6.3. *Let E be a semistable rational G -bundle which is not strongly semistable. Let us take l such that $\tilde{E} = (F^l)^*E$ has strong Harder–Narasimhan filtration and let \tilde{E}_P be the canonical reduction of \tilde{E} . Let μ be a fundamental weight of P and let Q be the corresponding maximal parabolic containing P . Then for some integer $0 \leq i < l$ we have*

$$0 < \mu(\tilde{E}_Q(V_\mu)) \leq \mu_{\max}((F^i)^*\Omega_X),$$

where \tilde{E}_Q is the extension of structure group of \tilde{E}_P .

Proof. Note that \tilde{E}_Q does not descend l times under the Frobenius morphism, since by Proposition 6.2 it would contradict semistability of E . Let i denote the non-negative integer such that \tilde{E}_Q descends i times under Frobenius, but it does not descend $(i+1)$ times. Let us write $\tilde{E}_Q = (F^i)^*E_Q$ for some reduction E_Q of $(F^{l-i})^*E$.

Let $\sigma: X \rightarrow (F^{l-i})^*E/Q$ be the section corresponding to the reduction E_Q . Then we have a map $\tau: T_X \rightarrow \sigma^*N_\sigma$, where N_σ is the normal bundle of $\sigma(X)$ in $(F^{l-i})^*E/Q$ (see, e.g., [MS], the proof of Theorem 4.1). Moreover, $\sigma^*N_\sigma = E_Q(\mathfrak{g}/\mathfrak{q})$ and the map τ is non-zero, because otherwise σ would descend under the Frobenius morphism, contradicting our assumption on i . Hence the map $(F^i)^*(T_X) \rightarrow \tilde{E}_Q(\mathfrak{g}/\mathfrak{q})$ is also non-zero. This implies that

$$\mu_{\min}((F^i)^*(T_X)) \leq \mu_{\max}(\tilde{E}_Q(\mathfrak{g}/\mathfrak{q})).$$

But $\mu_{\max}(\tilde{E}_Q(\mathfrak{g}/\mathfrak{q})) = -\mu_{\min}(\tilde{E}_Q(\mathfrak{u}))$, where \mathfrak{u} is the Lie algebra of the unipotent radical $R_u(Q)$ of Q . Hence we get

$$\mu_{\min}(\tilde{E}_Q(\mathfrak{u})) \leq \mu_{\max}((F^i)^*\Omega_X).$$

Note that $\tilde{E}_Q(\mathfrak{u})$ has a filtration with strongly semistable quotients, whose slopes are equal to multiples of $\mu(\tilde{E}_Q(V_\mu))$ (by Lemma 4.5) and $\tilde{E}_Q(V_\mu)$ is a quotient of $\tilde{E}_Q(\mathfrak{u})$. Therefore $\mu_{\min}(\tilde{E}_Q(\mathfrak{u})) = \mu(\tilde{E}_Q(V_\mu))$, Q.E.D.

Corollary 6.4. ([MS], Theorem 4.1) *If $\mu_{\max}(\Omega_X) \leq 0$ then every semistable rational G -bundle is strongly semistable. In particular, every rational G -bundle on X has strong Harder–Narasimhan filtration.*

Proof. If there exists a semistable G -bundle which is not strongly semistable then by Theorem 5.1 there also exists a semistable G -bundle E such that F^*E is not semistable, but it has strong Harder–Narasimhan filtration. But this contradicts Theorem 6.3, Q.E.D.

6.5. Let P be a parabolic subgroup of G . Then by [Be2], Proposition 1.9 there exist some positive numbers $b_{\mu,P}$ such that

$$\sum_{\alpha \in U(P)} \alpha = \sum_{\mu \in \text{vert } P} b_{\mu,P} \mu.$$

Corollary 6.6. *Assume that $\mu_{\max}(\Omega_X) > 0$. Let E be a semistable G -bundle and let P be the parabolic subgroup of G corresponding to the strong Harder–Narasimhan filtration of some Frobenius pull back of E . Then*

$$\deg_{\text{HN},\infty} E \leq \left(\sum_{\mu \in \text{vert } P} b_{\mu,P} \langle \mu, \mu^\vee \rangle \right) \frac{L_{\max}(\Omega_X)}{p}.$$

In particular, we have

$$\deg_{\text{HN}} E(\mathfrak{g}) \leq \deg_{\text{HN},\infty} E(\mathfrak{g}) \leq \frac{2 \dim \mathfrak{g}}{p} \cdot \left(\sum_{\mu \in \text{vert } P} b_{\mu,P} \langle \mu, \mu^\vee \rangle \right) L_{\max}(\Omega_X).$$

Proof. Let us take l such that both $(F^l)^*E$ and $(F^{l-1})^*(\Omega_X)$ have strong Harder–Narasimhan filtrations. Then $\mu_{\max}((F^{l-1})^*\Omega_X) = p^{l-1}L_{\max}(\Omega_X)$. Note that

$$\begin{aligned} \deg_{\text{HN},\infty} E &= \left\langle \sum_{\alpha \in U(P)} \alpha, y(P) \right\rangle = \sum_{\lambda \in \text{vert } P} \mu(E_P(V_\lambda)) \left\langle \sum_{\alpha \in U(P)} \alpha, \lambda^\vee \right\rangle \\ &= \sum_{\mu \in \text{vert } P} b_{\mu,P} \sum_{\lambda \in \text{vert } P} \mu(E_P(V_\lambda)) \langle \mu, \lambda^\vee \rangle. \end{aligned}$$

Hence the first inequality follows from Proposition 6.2 and Theorem 6.3. The second inequality follows from Corollary 3.4, Q.E.D.

Let us set

$$b(G) = 2 \dim \mathfrak{g} \cdot \max_{P \subset G} \left(\sum_{\mu \in \text{vert } P} b_{\mu,P} \langle \mu, \mu^\vee \rangle \right),$$

where the maximum is taken over a finite set of all parabolic subgroups of P containing B .

Corollary 6.7. *Assume that $\mu_{\max}(\Omega_X) > 0$ and $p > b(G) \cdot L_{\max}(\Omega_X)$. Then E is semistable if and only if $E(\mathfrak{g})$ is semistable.*

Proof. One implication follows from Corollary 2.8. The other implication follows from Corollary 6.6 and the remark that $\deg_{\text{HN}} E(\mathfrak{g})$ is an integer, Q.E.D.

Corollary 6.7 is similar to, but usually weaker than, the main result of [IMP] applied to the adjoint representation. However, Corollary 6.6 bounds the degree of instability of the adjoint bundle $E(\mathfrak{g})$ even in small characteristic, so in the cases when [IMP] gives no information. In the same way one can use Theorem 8.4 to prove that an extension of the structure group of a semistable G -bundle is semistable if the characteristic p is large. In the case when the corresponding group homomorphism is a representation we get a weak form of the main result of [IMP].

7. BOUNDEDNESS OF PRINCIPAL G -BUNDLES

7.1. An algebraic family of rational G -bundles on X parametrised by S is a rational G -bundle on $X \times S$ whose restriction to each fibre of the projection $X \times S \rightarrow S$

S is a rational G -bundle. A family \mathcal{E} of principal G -bundles on X is called *bounded* if there exists an algebraic family of principal G -bundles on X parametrised by a scheme of finite type over k and such that it contains each element of \mathcal{E} (up to an isomorphism).

7.2. Let us recall that the *degree* of a principal G -bundle E is a homomorphism $d_E: X^*(G) \rightarrow \mathbb{Z}$ given by $\chi \rightarrow \deg E(\chi)$, where $E(\chi)$ is the line bundle associated to E by χ .

Note that the character group $X^*(G)$ of G is a subgroup of finite index in $X^*(R(G))$ (this follows from the well known facts saying that $G = R(G) \cdot (G, G)$ and $R(G) \cap (G, G)$ is finite). Hence we can uniquely extend d_E to a homomorphism $X^*(R(G)) \rightarrow \mathbb{Q}$. This homomorphism should be thought of as a slope of E (look at the $G = \mathrm{GL}(V)$ case).

Since $R(G) \subset T$ we also get the induced homomorphism $X^*(T) \rightarrow X^*(R(G)) \rightarrow \mathbb{Q}$ denoted by d'_E . Note that

$$X^*(G) = X^*(T)_0 = \{\lambda \in X^*(T): \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Phi\}$$

and the restriction of $d'_E: X^*(T) \rightarrow \mathbb{Q}$ to $X^*(G) \subset X^*(T)$ induces the original degree homomorphism $d_E: X^*(G) \rightarrow \mathbb{Z}$.

We can interpret d'_E in the following way. Let

$$X^*(T)_+ = \{\lambda \in X^*(T): \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$$

denote the dominant weights of T with respect to the set of positive roots Φ^+ . For any $\lambda \in X^*(T)_+$ we denote by $L(\lambda)$ the simple G -module with highest weight λ . The radical $R(G)$ is contained in the centre $Z(G)$ and hence it acts on $L(\lambda)$ through the restriction of λ to $R(G)$ (see [Ja], Part II, 2.10). This implies that the radical $R(G)$ acts on $\det E(L(\lambda))$ through the restriction of $\lambda^{\dim L(\lambda)}$ to $R(G)$. Hence

$$d'_E(\lambda) = \mu(E(L(\lambda))).$$

Theorem 7.3. *Let us fix some constant C . Then the family \mathcal{F} of all semistable principal G -bundles E on X such that the degree of E is fixed and $a_2(\tilde{E}(\mathfrak{g})) \geq C$ (see 1.4), is bounded.*

Proof. By Corollary 6.6 and Proposition 2.4 the slopes of maximal destabilizing subsheaves of $\tilde{E}(\mathfrak{g})$ for $E \in \mathcal{F}$ are bounded from the above by some (explicit) constant C' . Then Theorem 1.5 implies that the family $\mathcal{F}' = \{E(\mathfrak{g})\}_{E \in \mathcal{F}}$ of rational vector bundles is bounded.

Let G' be the image of the adjoint representation $\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})$. It is equal to the quotient of G by the centre group scheme $Z(G)$. Let $\mathcal{E} = \{E_{G'}\}_{E \in \mathcal{F}}$ be the family of G' -bundles obtained from G -bundles in \mathcal{F} by extension of structure group to G' . Since $G' \hookrightarrow \mathrm{GL}(\mathfrak{g})$, each rational G' -bundle in \mathcal{E} can be constructed as a reduction of structure group of a rational $\mathrm{GL}(\mathfrak{g})$ -principal bundle F from \mathcal{F}' . But such reductions of structure group can be parametrized by a scheme of finite type corresponding to sections $U \rightarrow F(\mathrm{GL}(\mathfrak{g})/G')$ defined on some big open subset U of X . Since the family \mathcal{F}' is bounded, this shows that the family \mathcal{E} is also bounded (see [Re2] Lemma 4.8.1 for a precise argument).

For any character $\lambda \in X^*(T)$ let k_λ be a one-dimensional B -module whose restriction to T is λ . Let $H^0(\lambda)$ denote the G -module $H^0(G/B, \mathcal{L}_\lambda)$, where \mathcal{L}_λ is the line bundle associated to the B -module k_λ . Note that $H^0(\lambda)$ are usually not simple G -modules, but the centre group scheme $Z(G)$ acts on $H^0(\lambda)$ through scalars (see [Ja], Chapter II, Proposition 2.8). In fact, it acts through the restriction of λ to $Z(G)$.

Let $\rho: G \rightarrow \mathrm{GL}(V)$ be any finite-dimensional faithful rational representation, which is a direct sum of G -modules of the form $H^0(\lambda)$ for some $\lambda \in X^*(T)$. Then for any direct summand $H^0(\lambda)$ of V we have a group homomorphism $G' \rightarrow \mathrm{PGL}(H^0(\lambda))$ induced from ρ . Hence the family \mathcal{F}_λ of $\mathrm{PGL}(H^0(\lambda))$ -bundles obtained from the family \mathcal{F} by extension of structure group $G \rightarrow G' \rightarrow \mathrm{PGL}(H^0(\lambda))$ is also bounded.

Let \mathcal{E}_λ be the family of vector bundles associated to $\mathrm{GL}(H^0(\lambda))$ -bundles obtained from \mathcal{F} by extension of structure group $G \rightarrow \mathrm{GL}(H^0(\lambda))$. Two $H^0(\lambda)$ -vector bundles give the same principal $\mathrm{PGL}(H^0(\lambda))$ -bundle if and only if their projectivisations are isomorphic, i.e., they differ by tensoring by a line bundle. Since the degree in the family \mathcal{F} is fixed the degree of vector bundles in the family \mathcal{E}_λ is also fixed. This and the fact that the family \mathcal{F}_λ is bounded imply that the family \mathcal{E}_λ is also bounded (see [Ra3], 4.15 for a general argument).

Hence the family $\mathcal{E} = \{E(V)\}_{E \in \mathcal{F}}$ of vector bundles, obtained as direct sums of vector bundles from families \mathcal{E}_λ , is also bounded (see [Ra3], the proof of Proposition 4.12). But, as before, this implies that the family \mathcal{F} is also bounded, Q.E.D.

If in the above proof instead of Corollary 6.6 we use Corollary 8.5 (and Corollaries 2.8 and 6.4) then we get the following theorem:

Theorem 7.4. *Let C_1 and C_2 be fixed constants. Then the family of all rational G -bundles E on X such that the degree of the canonical parabolic of E is $\leq C_1$, the degree of E is fixed, and $a_2(\tilde{E}(\mathfrak{g})) \geq C_2$ is bounded.*

As a special case of the above theorem we get Theorem 0.1.

8. INSTABILITY OF BUNDLES ASSOCIATED TO REPRESENTATIONS

In this section we give an explicit bound for $L_{\max}(E(\mathfrak{h}))$ for a semistable G -bundle E and a homomorphism $\rho: G \rightarrow H$ of semisimple groups (see Theorem 8.4). In particular, we get the bound for the difference $\mu_{\max}(E(V)) - \mu_{\min}(E(V)) \leq L_{\max}(E(\mathfrak{gl}(V)))$ for an arbitrary representation $G \rightarrow \mathrm{GL}(V)$ of a semisimple group.

8.1. Let us first start with the simplest case of $G = \mathrm{SL}(V)$, where the bound is particularly strong.

Set $n = \dim V$ and let $T_{\mathrm{GL}(V)}$ be the standard maximal torus of $\mathrm{GL}(V)$. Let ϵ_i , $i = 1, \dots, n$ be the standard basis of $X^*(T_{\mathrm{GL}(V)})$, i.e., ϵ_i is the restriction of the matrix coefficient x_{ii} to $T_{\mathrm{GL}(V)}$. Let $T_{\mathrm{SL}(V)} = T_{\mathrm{GL}(V)} \cap \mathrm{SL}(V)$ be the corresponding maximal torus in $\mathrm{SL}(V)$. Then $X^*(T_{\mathrm{SL}(V)}) = X^*(T_{\mathrm{GL}(V)})/\mathbb{Z}(\epsilon_1 + \dots + \epsilon_n)$. Let $\omega_i = \epsilon_1 + \dots + \epsilon_i$ for $i = 1, \dots, n$ be the dominant weights of $\mathrm{GL}(V)$. Then the restrictions $\omega'_i = \omega_i|_{T_{\mathrm{SL}(V)}}$ for $i = 1, \dots, n-1$ are the dominant weights of $\mathrm{SL}(V)$.

Let W be a polynomial $\mathrm{GL}(V)$ -module. Then there exists m such that W is a submodule of $V^{\otimes m}$ (see [KP], Proposition 5.3). In particular, if $W = L(\lambda)$ is the simple $\mathrm{GL}(V)$ -module with highest weight $\lambda = \sum_{i=1}^n m_i \omega_i$, then m is uniquely determined by λ and it is equal to the degree $|\lambda| = \sum_{i=1}^n i m_i$. This follows from the facts that the scalars act on $L(\lambda)$ through the restriction of λ and $L(\lambda) = \bigwedge^i V^{\otimes m}$.

Every dominant weight of $\mathrm{SL}(V)$ can be written as a sum $\lambda = \sum_{i=1}^{n-1} m_i \omega'_i$ of fundamental weights. Then the corresponding weight $\lambda' = \sum_{i=1}^{n-1} m_i \omega_i$ of $\mathrm{GL}(V)$ is polynomial (see, e.g., [Ja], Proposition A.3). Hence by the above the $\mathrm{GL}(V)$ -module $L(\lambda')$ is a submodule of $V^{|\lambda'|}$. But $L(\lambda')$ is the simple $\mathrm{SL}(V)$ -module with highest weight $\lambda = \lambda'|_{T_{\mathrm{SL}(V)}}$ (see [Ja], II, 2.10.(2)).

Hence the simple $\mathrm{SL}(V)$ -module $L(\lambda)$ with highest weight λ is an $\mathrm{SL}(V)$ -submodule of $V^{\otimes |\lambda|}$, where $|\lambda| = \sum_{i=1}^{n-1} i m_i$ is the degree of λ .

Let W be an $\mathrm{SL}(V)$ -module. Then the maximum of degrees of fundamental weights, whose modules occur as quotients of the Jordan–Hölder filtration of W is called the JH-degree of W and denoted by $\mathrm{JH}(W)$.

Lemma 8.2. *Let W be an $\mathrm{SL}(V)$ -module. Let E be a principal $\mathrm{SL}(V)$ -bundle and let $E(V)$ and $E(W)$ denote the associated vector bundles. Then*

$$\mathrm{JH}(W) \cdot L_{\min}(E(V)) \leq L_{\min}(E(W)) \leq L_{\max}(E(W)) \leq \mathrm{JH}(W) \cdot L_{\max}(E(V)).$$

Proof. If W is not a simple $\mathrm{SL}(V)$ -module then take the Jordan–Hölder filtration of W and let V_i denote the quotients of this filtration. Since $L_{\max}(E(W))$ is less or equal to the maximum of $L_{\max}(E(V_i))$ and V_i are simple, we can assume that W is simple. Then W is isomorphic to some $L(\lambda)$. Since $L(\lambda)$ is a submodule of $V^{\otimes |\lambda|}$, so $E(W)$ is a subbundle of $E(V)^{\otimes |\lambda|}$. Therefore by 1.2 we have

$$L_{\max}(E(W)) \leq |\lambda| \cdot L_{\max}(E(V)).$$

The second inequality follows from the above one applied to the dual representation, Q.E.D.

Corollary 8.3. *Let $\rho: \mathrm{SL}(V) \rightarrow \mathrm{GL}(W)$ be a homomorphism and let E be a principal $\mathrm{SL}(V)$ -bundle. Then*

$$L_{\max}(E(\mathfrak{gl} W)) = L_{\max}(E(W)) - L_{\min}(E(W)) \leq \mathrm{JH}(\rho') \cdot L_{\max}(E(V)),$$

where $\rho' = \mathrm{Ad}_{\mathrm{GL}(W)} \circ \rho$.

In particular, if E is semistable, $\mathrm{char} k = p$ and $\mu_{\max}(\Omega_X) > 0$ then

$$L_{\max}(E(\mathfrak{gl} W)) \leq (\dim V - 1) \mathrm{JH}(\rho') \cdot \frac{L_{\max}(\Omega_X)}{p}.$$

Proof. The first inequality follows from Lemma 8.2 applied to the representation ρ' . The second inequality follows from the first one and Theorem 1.3, Q.E.D.

We can apply a similar method to prove a theorem similar to the second part of Corollary 8.3 for any homomorphism of semisimple groups (in fact, the statement is slightly more general):

Theorem 8.4. *Let $\rho: G \rightarrow H$ be a homomorphism of connected reductive groups over k . Assume that $\rho(R(G)) \subset R(H)$. Let E_G be a semistable principal G -bundle and let E_H be the extension of structure group of E_G to H .*

(1) *If $\mathrm{char} k = 0$ or $\mu_{\max}(\Omega_X) < 0$ then E_H is strongly semistable.*

- (2) If $\text{char } k = p$ and $\mu_{\max}(\Omega_X) > 0$ then there exists some explicit constant $C(\rho)$ depending only on ρ such that

$$0 \leq L_{\max}(E_H(\mathfrak{h})) \leq C(\rho) \cdot \frac{L_{\max}(\Omega_X)}{p}.$$

In particular, if p is large then both $E_H(\mathfrak{h})$ and E_H are semistable.

Proof. (1) follows from Theorem 2.7 and Corollary 6.4. Hence we can assume that we are in case (2).

Let $\lambda: G_m \rightarrow G$ be a 1-parameter subgroup of G . Then we can associate to λ a closed subgroup $P(\lambda)$ of G by

$$P(\lambda) = \{p \in G: \lim_{t \rightarrow 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1} \text{ exists in } G\}.$$

It is a parabolic subgroup and any parabolic subgroup P of a reductive group G is of this form for some 1-parameter subgroup λ (see [Sp], Proposition 8.4.5). The unipotent radical $R_u(P(\lambda))$ of $P(\lambda)$ consists of such points $p \in P(\lambda)$ that $\lim_{t \rightarrow 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1} = e$, where e is the neutral element in G .

By Theorem 5.1 we can take such l that $\tilde{E} = (F^l)^*E$ has the strong canonical reduction \tilde{E}_P . Let λ be a 1-parameter subgroup such that P is associated to λ and let Q be the parabolic subgroup of H associated to $\rho \circ \lambda$. Then $\rho(P) \subset Q$ and $\rho(R_u(P)) \subset R_u(Q)$. There exists a filtration of \mathfrak{h} with simple Q -modules as quotients and such that $R_u(Q)$ acts trivially on each factor. This can be constructed by taking $\mathfrak{u} \subset \mathfrak{q} \subset \mathfrak{h}$, where \mathfrak{u} is the Lie algebra of $R_u(Q)$, and taking the corresponding filtrations of \mathfrak{u} , $\mathfrak{q}/\mathfrak{u}$ and $\mathfrak{h}/\mathfrak{q}$ (see 1.1). Now take a further refinement of this filtration $V_m \subset V_{m-1} \subset \dots \subset V_0 = \mathfrak{h}$ such that the quotients are simple P -modules. By construction $R_u(P)$ acts trivially on each quotient $W_i = V_i/V_{i+1}$ of this filtration and hence $\tilde{E}_H(V_i)/\tilde{E}_H(V_{i+1}) = \tilde{E}_L(W_i)$. Since W_i is a simple L -module, by Schur's lemma the radical of L (which is contained in $Z(L)$) acts on W_i by scalars. In particular, the above filtration gives rise to a filtration of $\tilde{E}_H(\mathfrak{h})$ with strongly semistable quotients. Degrees of these quotients can be determined in the following way. Note that $P \rightarrow P/R_u(P) = L$ induces the map $X^*(L) \rightarrow X^*(P)$ of character groups, which we compose with the degree map $d_{\tilde{E}_P}: X^*(P) \rightarrow \mathbb{Z}$. In this way we get the degree map $d_{\tilde{E}_L}: X^*(L) \rightarrow \mathbb{Z}$. As in 7.2 we can extend it to $d_{\tilde{E}_L}: X^*(T_L) \rightarrow \mathbb{Q}$, where T_L is a maximal torus in L . If W_i is a simple L -module with highest weight $\lambda_i \in X^*(T_L)$, then the slope of $\tilde{E}_L(W_i)$ can be computed as $d_{\tilde{E}_L}(\lambda_i)$. Writing λ_i as a sum of fundamental weights of L , we can use Proposition 6.2 and Theorem 6.3 (or Corollary 6.6), to bound $d_{\tilde{E}_L}(\lambda_i)$ by means of the coefficients in the sum times $p^{l-1}L_{\max}(\Omega_X)$. In particular, since

$$L_{\max}(E(\mathfrak{h})) = \frac{\mu_{\max}(\tilde{E}(\mathfrak{h}))}{p^l} \leq \max_i \frac{\mu(\tilde{E}_L(W_i))}{p^l},$$

this gives the required explicit bound on $L_{\max}(E(\mathfrak{h}))$, Q.E.D.

Remarks.

(1) Note that the above theorem also bounds $\deg_{\text{HN}} E_H$. This follows from the definition, since $\deg_{\text{HN}} E_H$ is the degree of a subbundle of the degree zero vector bundle $E_H(\mathfrak{h})$.

(2) From the proof of the above theorem one can easily see that $C(\rho)$ can be explicitly bounded by means of the heights of the composition factors of the induced L -module \mathfrak{h} , where L is the Levi component of some parabolic subgroup of G containing a fixed maximal torus T .

Corollary 8.5. *Assume that $\text{char } k = p$ and $\mu_{\max}(\Omega_X) > 0$. There exists a constant B_G depending only on G such that for every principal G -bundle E we have*

$$\mu_{\max}(E(\mathfrak{g})) \leq \deg_{\text{HN}} E + B_G \cdot \frac{L_{\max}(\Omega_X)}{p}.$$

Proof. Let E_P be the Harder–Narasimhan filtration of E and let $L = P/R_u(P)$ be the Levi subgroup of P . Since $E(\mathfrak{g})$ has a filtration with quotients $E_L(V_S)$ for all possible shapes S , we have

$$(*) \quad \mu_{\max}(E(\mathfrak{g})) \leq \max_S \mu_{\max}(E_L(V_S)).$$

Now let us recall that

$$\deg_{\text{HN}} E = \deg E_P(\mathfrak{p}) = \sum_{l(S) \geq 0} \deg E_L(V_S)$$

and $\deg E_L(V_S)$ are non-negative if $l(S) \geq 0$. Hence for any shape S we have $|\deg E_L(V_S)| \leq \deg_{\text{HN}} E$. Since E_L is semistable, Theorem 8.4 implies that there exists a constant $C_L(V_S)$ such that

$$\begin{aligned} \mu_{\max}(E_L(V_S)) - \mu_{\min}(E_L(V_S)) &\leq L_{\max}(E_L(V_S)) - L_{\min}(E_L(V_S)) \\ &= L_{\max}(E_L(\mathfrak{gl}(V_S))) \leq C_L(V_S) \cdot \frac{L_{\max}(\Omega_X)}{p}. \end{aligned}$$

Since if we fix a maximal torus there are only finitely many possible choices for P and L , it follows that there exists B_G such that $C_L(V_S) \leq B_G$ for all possible P , L and S . Then

$$\mu_{\max}(E_L(V_S)) \leq \mu(E_L(V_S)) + B_G \cdot \frac{L_{\max}(\Omega_X)}{p} \leq \frac{\deg_{\text{HN}} E}{\dim V_S} + B_G \cdot \frac{L_{\max}(\Omega_X)}{p},$$

which by $(*)$ implies the required inequality, Q.E.D.

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